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Analyzing Root Structures of Generalized Kac-Moody Algebra G(3,3)

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ABSTRACT

In the paper, we consider the GKM algebras associated with the Generalized Generalized Cartan matrices (GGCM) which are extensions of some irreducible highest weight modules $\mathrm{V}(\lambda)$ with highest weight λ over the GKM algebra $\mathrm{g} = \mathrm{g}(3,3)$ with $\lambda(\mathrm{h}_0) = 0, \lambda(\mathrm{h}_1) = 3$. For this family, we compute the root multiplicities of all roots using Witt partition function.

***Keywords**: Dimension, Partition function, Hyperbolic, Root Multiplicities*

I. INTRODUCTION

In recent years the area of infinite-dimensional Lie algebras has attracted considerable attention because of its numerous connections with other topics in mathematics and, not least, its importance in theoretical physics. Borcherds introduced the concept of generalized Kac-Moody algebras (GKM algebras) in [2]. GKM algebras differ from Kac-Moody algebras in that they may possess simple imaginary roots. Determining the multiplicities for imaginary roots is still a crucial problem. In [5], Kim and Shin computed the recursion dimension formula for all graded Lie algebras. A closed form root multiplicity formula, for all the roots of GKM algebras has been derived in [6] and [7]. Root multiplicities for the indefinite kac-Moody algebras $HD_4^{(3)}$, $HG_2^{(1)}$ and $HD_n^{(1)}$ were determined in [3] and [4]. The classification of purely imaginary, Strictly imaginary and special imaginary roots were delimited in [8], [9], [10], [11], [12]. Later, in [13] and [14], determined the root structure for the family EB_2.

In this paper, by connecting the above results, we determine the root multiplicity of GKM algebras associated with the Generalized Generalized Cartan matrices(GGCM) which are extensions of some irreducible highest weight modules $V(\lambda)$ with highest weight λ over the GKM algebra $g = g(3,3)$ with $\lambda(h_0) = 0, \lambda(h_1) = 3$.

II. PRELIMINARIES

The definitions and notations are as in [2], [5], [15] and [16].
Let $I = \{1,2,$â be a finite or countably infinite index set and $A = (a_{i,j})_{i,j)} \in I$ be a real matrix satisfying the following conditions:

1. either $a_{ii} = 2$ or $a_{ii} \leq 0 \forall i \in I$;
2. $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ij} = 2$;
3. $a_{ij} = 0$ implies $a_{ij} = 0$.

A is called a generalized generalized Cartan matrix and the Lie algebra $g(A)$ associated with A is called the generalized Kac-Moody algebra. We assume that a GGCM A is symmetrizable if $\exists$ a diagonal matrix $D = diag(s_i | i \in I)$ with $s_i > 0 (i \in I)$ such that DA is symmetric.

Let $I^{re} = \{i \in I | a_{ii} = 2\}$, $I^{im} = \{i \in I | a_{ii} \leq 0\}$, and let $\underline{m} = (m_i \in \mathbb{Z}_{>0} | i \in I)$ be a collectioon of positive integers such that $m_i = 1$ for all $i \in I^{re}$.

The GKM algebra $g = g(A, \underline{m})$ associated with a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$ of charge $\underline{m} = (m_i | i \in I)$ is the Lie algebra generated by the elements $h_i, d_i (i \in I), e_{ik}, f_{ik} (i \in I, k = 1 \cdots, m_i)$ with the defining relations:

$[h_i, h_j] = [d_i, d_j] = [h_i, d_j] = 0$, $\quad [h_i, e_{jl}] = a_{ij} e_{jl}, [h_i, f_{jl}] = -a_{ij} f_{jl}$,
$[d_i, e_{jl}] = \delta_{ij} e_{jl}, [d_i, f_{jl}] = -\delta_{ij} f_{jl}$, $\quad [e_{ik}, f_{jl}] = \delta_{ij} \delta_{kl} h_i$,

$(ade_{ik})^{1-a_{ij}}(e_{jl}) = (adf_{ik})^{1-a_{ij}}(f_{jl}) = 0$ if $a_{ii} = 2, i \neq j$,
$[e_{ik}, e_{jl}] = [f_i k, f_{jl} = 0$ if $a_{ij} = 0$ where, $(i, j \in I, k = 1, \cdots, m_i, l = 1, \cdots, m_j)$.

Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, and $Q^- = -Q^+$. The GKM algebra $g = g(A, \underline{m})$ has the root space deccomposition,
$g(A) = \bigoplus_{\alpha \in Q} g_\alpha$, where $g_\alpha = \{x \in g | [h, x] = \alpha(h)x$, for all $h \in H\}$

An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$. The number $mult\alpha = \dim g_\alpha$ is called the multiplicity of the root α. Note that $mult\alpha_i = mult(-\alpha_i) = m_i$ for all $i \in I$.

Let $P^+ = \{\lambda \in h^* | \lambda(h_i) \geq 0$ for all $i \in I$, $\lambda(h_i) \in \mathbb{Z}_{\geq} 0$ if $a_{ii} = 2\}$, and let $V(\lambda)$ be the irreducible highest weight module over g with highest weight λ.

Let J be a finite subset of I^{re} and we denote by $\Delta_J = \Delta \cap (\sum_{j \in J} \mathbb{Z}\alpha_i), \Delta_J^\pm = \Delta_J \cap \Delta^\pm$ and $\Delta^\pm(J) = \Delta^\pm \backslash \Delta_J^\pm$. We also denote by $Q_J = Q \cap (\sum_{j \in J} \mathbb{Z}\alpha_i), Q_J^\pm = Q_J \cap Q^\pm$ and $Q^\pm(J) = Q^\pm \backslash Q_J^\pm$. Define $g_0^{(}J) = h \oplus (\bigoplus_{\alpha \in \Delta_J} g_\alpha)$ and $g_\pm^{(}J) = \bigoplus_{\alpha \in \Delta_J^\pm} g_\alpha$. Thus we have the triangular decomposition: $g = g_-^{(J)} \oplus g_0^{(J)} \oplus g_+^{(J)}$, where $g_0^{(J)}$ is the Kac-Moody algebra associated with the generalized Cartan matrix $A_J = (a_{ij})_{i,j \in J} and g_-^{(J)} (resp. g_+^{(J)}$ is a direct sum of irreducible heighest weight (resp. lowest weight) modules over $g_0^{(J)}$.

Let $W_J = \langle r_j | j \in J \rangle$ be the subgroup of W generated by the simple reflections $r_j (j \in J)$ and let $W(J) = \{w \in W | w\Delta^- \cap \Delta^+ \subset \Delta^+(J)\}$. Then W_J is the Weyl group of the Kac-Moody algebra $g_0^{(J)}$ and $W(J)$ be the set of right coset representatives of W_J in W.

Proposition: $H_K^{(J)} = \bigoplus_{\substack{w \in W(J) \\ F \subset T \\ l(w) + |F| = k}} V_J(w(\rho - s(F)) - \rho)$ where $V_J(\mu)$ denotes the irreducible highest weight module over $g_0^{(J)}$ with heighest weight μ and F runs over all the finite subsets of T such that any two elements in F are mutually perpendicular. Here we denote by $|F|$, the number of elements in F and $s(F)$, the sum of elements in F.
Define the homology space $H^{(J)}$ of $g_-^{(J)}$ to be

$$H^{(J)} = \sum_{k=1}^{\infty} (-1)^{k+1} H_k^{(J)} = \sum_{\substack{w \in W(J) \\ F \subset T \\ l(w) + |F| \geq 1}} (-1)^{l(w) + |F| + 1} V_J(w(\rho - s(F)) - \rho).$$

Let $P(H^{(J)}) = \{\alpha \in Q^-(J) | dim H_\alpha^{(J)} \neq 0\} = \{\tau_1, \tau_2, \tau_3, \tau_4, \cdots\}$
and $d(i) = din H_{\tau_i}^{(J)}$ for $i - 1,2, \cdots$. For $\tau \in Q^-(J)$,
we denote by $T^{(J)}(\tau)$ the set of all partitions of τ into a sum of τ_i,s, (i.e).,

$$T^{(J)}(\tau) = \{n = (n_i)_{i \geq 1} | n_i \in \mathbb{Z}_{\geq 0}, \sum n_i \tau_i = \tau\}.$$

For $n \in T^{(J)}(\tau)$, we will use the notaion $|n| = \sum n_i$ and $n! = \prod \ n_i!$. Now, for $\tau \in Q^-(J)$, we define a function

$$W^{(J)}(\tau) = \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \prod d(i)^{n_i}. \qquad (1)$$

The function $W^{(J)}(\tau)$ is called the Witt partition function.

Theorem 1: Let $\alpha \in \Delta^-(J)$ be a root of a symmetrizable GKM algebra g. Then we have

$$dim g_\alpha = \sum_{d|\alpha} \frac{1}{d} \mu(d) W(J)(\tfrac{\alpha}{d})$$
$$= \sum_{d|\alpha} \frac{1}{d} \mu(d) \sum_{n \in T^{(J)}(\frac{\alpha}{d})} \frac{(|n|-1)!}{n!} \prod d(i)^{n_i}$$

where μ is the classical Mobius function.

III. ROOT MULTIPLICITY OF SOME GKM ALGEBRA $\mathbf{g} = \mathbf{g}(3,3)$

In this section, we explicitly determine the root multiplicities of GKM algebra associated with the GGCM which is an extension of some irreducible highest weight modules $V(\lambda)$ with highest weight λ over the GKM algebra $g = g(3{,}3)$ with $\lambda(h_0) = 0, \lambda(h_1) = 3$.

3.1 Consider the GKM algebra $g = g(A, \underline{m})$ associated with the Generalized Generalized Cartan Matrix (GGCM)

$$\begin{pmatrix} -k & 0 & -3 \\ 0 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix}$$

of charge $\underline{m} = (s, 1{,}1)$ where $k, s \in \mathbb{Z}_{>0}$. This matrix is an extension of HA_1.

Let $I = \{1{,}2{,}3\}$ be the index set for the simple roots of g. Then, α_1 is the imaginary simple root with multiplicity $r \geq 1$ and α_2, α_3 are the real simple roots.

Then we have
$T = \{\alpha_1, \alpha_1, \cdots, \alpha_1\}$ counted s times.
Since $(\alpha_1, \alpha_1) = -k < 0$, F can be either empty or $\{\alpha_1\}$.
If we take $J = \{2{,}3\}$, then
$g_0^{(J)} = g_0 \oplus \mathbb{C}h_1$,
where $g_0 = \langle e_2, f_2, h_2, e_3, f_3, h_3 \rangle$ and $W(J) = \{1\}$

By proposition, we have
$H_1^{(J)} = V_J(-\alpha_1) \oplus \cdots \oplus V_J(-\alpha_1)$ (s copies)
$H_2^{(J)} = 0$
$\vdots$
$H_k^{(J)} = 0$ for $k \geq 2$.

Therefore we get
$H^{(J)} = V_J(-\alpha_1) \oplus \cdots \oplus V_J(-\alpha_1)$ (s copies),
where $V_J(-\alpha_1)$ is the standard representation of A_4.
By identifying $-l_1\alpha_1 - l_2\alpha_2 - l_3\alpha_3 \in Q^-$ with $(l_1, l_2, l_3) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we have

$$P(H^{(J)}) = \{\tau_1, \tau_2, \tau_3, \tau_4, \cdots\}$$

where $\tau_1 = (1{,}0{,}0)$, $\tau_2 = (1{,}1{,}1)$, $\tau_3 = (1{,}0{,}1)$, $\tau_4 = (1{,}1{,}3)$, $\tau_5 = (1{,}1{,}2)$, $\tau_6 = (1{,}1{,}4)$, $\tau_7 = (1{,}2{,}1)$, $\cdots$

For the sake of completeness, we write down the Table 6.3 in [5].

m\n	0	1	2	3	4	5	6	7
0	1	1	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0
2	0	1	2	3	4	4	3	2
3	0	1	3	6	10	14	16	16
4	0	1	4	10	20	34	49	62
5	0	0	3	13	33	67	115	173
6	0	0	2	14	46	112	227	397

Using the above table, we can compute $\dim(V_J(-\alpha_1))_{\tau_i}$.
Every root of g is of the form (l_1, l_2, l_3) for $l_1 \geq 1$ and $l_2, l_3 \geq 0$. Here

$$d(i) = \dim(H^{(J)})_{\tau_i} = r\dim V_J(-\alpha_1)_{\tau_i}.$$

Thus, the Witt partition function (1) becomes

$$W^{(J)}(\tau) = \sum_{n \in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \dim V_J(-\alpha_1)_{\tau_i}.$$

Therefore, by the Theorem (1), we obtain the following proposition:

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Proposition : Let $g = g(A, \underline{m})$ be the GKM algebra associated with the GGCM $\begin{pmatrix} -k & 0 & -3 \\ 0 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix}$ of charge $\underline{m} = (s, 1,1)$ with $k, s \in \mathbb{Z}_{>0}$.
Thus, for the root $\alpha = -l_1\alpha_1 - l_2\alpha_2 - l_3\alpha_3$ with $l_1, l_2, l_3 \in \mathbb{Z}_{\geq 0}$, we have

$$\dim g_\alpha = \sum_{d|\alpha} \frac{1}{d}\mu(d) \sum_{n\in T^{(J)}(\tau)} \frac{(|n|-1)!}{n!} \dim V_J(-\alpha_1)_{\tau_i}. \qquad (2)$$

Example 3.1: Consider the root $\alpha = (2,4,6)$.

In the following table we gve the weights of $H^{(J)}$ and their multiplicities for $\lambda(h_0) = 0, \lambda(h_1) = 3$:

Weight	Multiplicity	Weight	Multiplicity	Weight	Multiplicity	Weight	Multiplicity
(1,0,0)	1	(1,5,2)	2	(1,3,4)	18	(1,2,6)	6
(1,0,1)	1	(1,0,3)	1	(1,4,4)	29	(1,3,6)	28
(1,1,1)	1	(1,1,3)	3	(1,5,4)	38	(1,4,6)	77
(1,2,1)	1	(1,2,3)	6	(1,6,4)	42	(1,5,6)	162
(1,3,1)	1	(1,3,3)	10	(1,1,5)	1	(1,6,6)	275
(1,0,2)	1	(1,4,3)	12	(1,2,5)	8	(1,2,7)	3
(1,1,2)	2	(1,5,3)	12	(1,3,5)	25	(1,3,7)	25
(1,2,2)	3	(1,6,3)	10	(1,4,5)	53	(1,4,7)	92
(1,3,2)	4	(1,1,4)	2	(1,5,5)	89	(1,5,7)	242
(1,4,2)	3	(1,2,4)	8	(1,6,5)	123	(1,6,7)	499

$T^{(J)}(2,4,6)$ corresponds to the partition of (2,4,6) into two parts. Therefore, the partitions of the root (2,4,6) into weights of $H^{(J)}$ are given in the following table:

(1,0,0)	(1,4,6)
(1,0,1)	(1,4,5)
(1,0,2)	(1,4,4)
(1,0,3)	(1,4,3)
(1,1,1)	(1,3,5)
(1,1,2)	(1,3,4)
(1,1,3)	(1,3,3)
(1,1,4)	(1,3,2)
(1,1,5)	(1,3,1)
(1,2,1)	(1,2,5)
(1,2,2)	(1,2,4)
(1,2,3)	(1,2,3)

Therefore, by the formula (2), for the root (2,4,6), we have

$$\dim g_{(2,4,6)} = 331r^2 - 6r$$

Example 3.2: Consider the root $\alpha = (2,2,3)$.
Similarly, $T^{(J)}(2,2,3)$ corresponds to the partition of (2,2,3) into two parts. Therefore, the partitions of the root (2,2,3) into weights of $H^{(J)}$ are given in the following table:

(1,0,0)	(1,2,3)
(1,0,1)	(1,2,2)
(1,0,2)	(1,2,1)
(1,1,1)	(1,1,2)

Therefore, by the formula (2), for the root (2,2,3), we have

$$\dim g_{(2,2,3)} = 12r^2$$

Example 3.3: Consider the root $\alpha = (3,3,3)$.
Similarly, $T^{(J)}(3,3,3)$ corresponds to the partition of (3,3,3) into three parts. Therefore, the partitions of the root (3,3,3) into weights of $H^{(J)}$ are given in the following table:

(1,0,0)	(1,0,0)	(1,3,3)
(1,0,0)	(1,0,1)	(1,3,2)
(1,0,1)	(1,0,1)	(1,3,1)
(1,0,1)	(1,1,1)	(1,2,1)
(1,1,1)	(1,1,1)	(1,1,1)
(1,0,0)	(1,1,2)	(1,2,1)
(1,0,0)	(1,0,2)	(1,3,1)
(1,0,0)	(1,1,1)	(1,2,2)

Therefore, by the formula (2), for the root (3,3,3), we have

$$\dim g_{(3,3,3)} = \frac{97r^3 - r}{3}$$

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