

Exploring Projective Relations of Two (α, β) -Metric Subclasses

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ABSTRACT

In this paper, we find the necessary and sufficient condition to characterize the projective relation between two subclasses of (α, β) -metrics $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ and $\bar{L} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms.

Keywords: Finsler space, (α, β) -metric, Kropina metric, Projective change, Douglas metric.

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I. INTRODUCTION

In Finsler geometry, two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if $G^i = \bar{G}^i + P y^i$, where G^i and $\bar{G}^i$ are the geodesic coefficients of F and $\bar{F}$ respectively and $P = P(x, y)$ is a scalar function on the slit tangent bundle TM_0 . In this case, any geodesic of the first is also geodesic for the second and viceversa. The projective changes between two Finsler spaces have been studied by [1], [2], [3], [4], [6], [11],[13],[14], [18], [19], [20].

(α, β) -metrics form a special and very important classes of Finsler metrics which can be expressed in the for $F = \alpha\varphi(s)$: $s = \frac{\beta}{\alpha}$, where α is a Riemannian metric and β is a 1-form and φ is a C^∞ positive function on the definite domain. In particular, when $\varphi = \frac{1}{s}$, the Finsler metric $F = \frac{\beta^2}{\alpha}$ is called Kropina metric. Kropina metric was first introduced by L. Berwald in connection with two dimensional Finsler space with rectilinear extremal and was investigated by V.K. Kropina [7]. They together with Randers metric are C-reducible [10]. However, Randers metric are regular Finsler metric but Kropina metric is non-regular Finsler metric. Kropina metric seem to be among the simplest nontrivial Finsler metric with many interesting applications in physics, electron optics with a magnetic field, dissipative mechanics and irreversible thermodynamics [5], [15]. Also, there are interesting applications in relativistic field theory, evolution and developmental biology.

Based on Stavrino's work on Finslerian structure of anisotropic gravitational field [16], we know that the anisotropy is an issue of the background radiation for all possible (α, β) -metrics. Then the 1-form β represents the same direction of the observed anisotropy of the microwave background radiation. That is, if two (α, β) -metrics $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ and $\bar{F} = \bar{\alpha}\bar{\varphi}\left(\frac{\bar{\beta}}{\bar{\alpha}}\right)$ are the same anisotropy directions (or, they have the same axis rotation to their indicatrices), then their 1-form.

β and $\bar{\beta}$ are collinear, there is a function $\mu \in C^\infty(M)$ such that $\beta(x, y) = \mu\bar{\beta}(x, y)$. By [3], for the projective equivalence between a general (α, β) -metric and a Kropina metric, we have the following lemma:

Lemma 1.1. Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be an (α, β) -metric on n -dimensional manifold M ($n \geq 3$), satisfying that β is not parallel with respect to α , $db \neq 0$ everywhere (or) $b = \text{constant}$ and F is not of Randers type. Let $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on the manifold M , where $\bar{\alpha} = \lambda(x)\alpha$ and $\bar{\beta} = \mu(x)\beta$. Then F is Projectively Equivalent to $\bar{F}$ if and only if the following equations holds,

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'), \quad (1.1)$$

$$G_\alpha^i = \bar{G}_\alpha^i + \theta y^i - \sigma(k_1 \alpha^2 + k_2 \beta^2)b^i, \quad (1.2)$$

$$b_{ij} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j], \quad (1.3)$$

$$\bar{s}_{ij} = \frac{1}{b^2}(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i), \quad (1.4)$$

where $\sigma = \sigma(x)$ is a scalar function and θ is 1-form, k_1, k_2, k_3 are constants. In this case, both $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

The purpose of this paper is to study the projective relation of two subclasses of (α, β) -metric. The main results of the paper are as follows.

Theorem 1.1. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an (α, β) -metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n -dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if they are Douglas metrics and the geodesic co-efficient of α and $\bar{\alpha}$ have the following relations

$$G_{\alpha}^i - 2\alpha^2 \tau b^i = \bar{G}_{\bar{\alpha}}^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \quad (1.5)$$

Where $b^i = a^{ij} b_j$, $\bar{b}^i = \bar{a}^{ij} \bar{b}_j$, $\bar{b}^2 = \|\bar{\beta}\|_{\bar{\alpha}}^2$ and $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on M

By [8] and [9], we obtain immediately from theorem (1.1), that

Proposition 1. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ an (α, β) -metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on a n -dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two nonzero collinear 1-forms. Then F is projectively equivalent to $\bar{F}$ if and only if the following equations hold:

$$G_{\alpha}^i - 2\alpha^2 \tau b^i = \bar{G}_{\bar{\alpha}}^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \quad (1.6)$$

$$b_{ij} = 2\tau\{(1 - 2b^2)a_{ij} + 3b_i b_j\}, \quad (1.7)$$

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i), \quad (1.8)$$

where b_{ij} denote the coefficient of the covariant derivatives of β with respect to α .

II. PRELIMINARIES

We say that a Finsler metric is projectively related to another Finsler metric if they have the same geodesic as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [2],

$$G_{\alpha}^i = G_{\bar{\alpha}}^i + \lambda_{x^k} y^k y^i, \quad (2.1)$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^i) denotes the local coordinates in the tangent bundle TM .

Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if and only if their spray coefficients have the relation [2],

$$G^i = \bar{G}^i + P(y)y^i \quad (2.2)$$

where $P(y)$ is a scalar function on $TM \setminus \{0\}$ and homogeneous of degree one in y .

For a given Finsler metric $L = L(x, y)$, the geodesic of L satisfy the following ODE:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

Where $G^i = G^i(x, y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4} g^{il} \{[F^2]_{x^m y^l y^m} - [F^2]_{x^l}\}.$$

Let $\varphi = \varphi(s)$, $|s| < b_0$, be a positive C^∞ function satisfying the following

$$\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0, \quad (|s| \leq b < b_0). \quad (2.3)$$

If $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is 1-form satisfying $\|\beta_x\|_{\alpha} < b_0 \forall x \in M$, then $F = \alpha\varphi(s)$,

$s = \frac{\beta}{\alpha}$, is called an (regular) (α, β) -metric. In this case, the fundamental form of the metric tensor induced by F is positive definite.

Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be covariant derivative of β with respect to α .
Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}); \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Note that β is closed if and only if $s_{ij} = 0$ [17].

Let $s_j = b^i s_{ij}$, $s_j^i = a^{il} s_{lj}$, $s_0 = s_i y^i$, $s_0^i = s_j^i y^j$ and $r_{00} = r_{ij} y^i y^j$.

The relation between the geodesic coefficients G^i of F and geodesic coefficients G_α^i of α is given by

$$G^i = G_\alpha^i + \alpha Q s_0^i \{-2Q\alpha s_0 + r_{00}\} + \Psi b^i + \theta \alpha^{-1} y^i, \quad (2.4)$$

Where

$$\begin{aligned} \theta &= \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}} \\ Q &= \frac{\varphi'}{\varphi - s\varphi'} \\ \Psi &= \frac{1}{2} \frac{\varphi''}{\{(\varphi - s\varphi') + (b^2 - s^2)\varphi''\}} \end{aligned}$$

For a Kropina metric $F = \frac{\alpha^2}{\beta}$, it is very easy to see that it is not a regular (α, β) -metric but the relation $\varphi(s) - s\varphi'(s) + (b^2 - s^2)\varphi''(s) > 0$ is still true for $|s| > 0$.

In [8], the authors characterized the (α, β) -metrics of Douglas type.

Lemma 2.2. [8]: Let $F = \alpha\varphi\left(\frac{\beta}{\alpha}\right)$ be a regular (α, β) -metric on an n -dimensional manifold M ($n \geq 3$). Assume that β is not parallel with respect to α and $db \neq 0$ everywhere or $b = \text{constant}$ and F is not of Randers type. Then F is a Douglas metric if and only if the function $\varphi = \varphi(s)$ with $\varphi(0) = 1$ satisfies the following ODE's

$$[1 + (k_1 + k_2 s^2)s^2 + k_3 s^2]\varphi'' = (k_1 + k_2 s^2)(\varphi - s\varphi'), \quad (2.5)$$

and β satisfies

$$b_{i|j} = 2\sigma[(1 + k_1 b^2)a_{ij} + (k_2 b^2 + k_3)b_i b_j] \quad (2.6)$$

Where $b^2 = \|\beta\|_\alpha^2$ and $\sigma = \sigma(x)$ is a scalar function and k_1, k_2, k_3 are constants $(k_2, k_3) \neq (0, 0)$.

For a Kropina metric, we have the following,

Lemma 2.3.[9]: Let $F = \frac{\alpha^2}{\beta}$ be Kropina metric on an n -dimensional manifold M . Then

(i) ($n \geq 3$) Kropina metric F with $b^2 \neq 0$ is Douglas metric if and only if

$$s_{ik} = \frac{1}{b^2}(b_i s_k - b_j s_i). \quad (2.7)$$

(ii) ($n = 2$) Kropina metric F is a Douglas metric.

Definition 2.1. [2]: Let

$$D_{jkl}^i = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right) \quad (2.8)$$

Where G^i is the spray coefficients of F . The tensor $D = D_{jkl}^i \partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

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We know that the Douglas tensor is a projective invariant [12]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes from (2.8). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{F}$.

Now, first we compute the Douglas tensor of a general (α, β) -metric.

Let

$$\hat{G}^i = G_\alpha^i + \alpha Q s_0^i + \Psi \{-2Q\alpha s_0 + r_{00}\} b^i, \quad (2.9)$$

then (2.4) becomes

$$G^i = \hat{G}^i + \theta \{-2Q\alpha s_0 + r_{00}\} \alpha^{-1} y^i.$$

Clearly, G^i and $\hat{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.

Let

$$T^i = \alpha Q s_0^i + \Psi \{-2Q\alpha s_0 + r_{00}\} b^i. \quad (2.10)$$

Then $\hat{G}^i = G_\alpha^i + T^i$, thus

$$\begin{aligned} D_{jkl}^i &= \hat{D}_{jkl}^i \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G_\alpha^i - \frac{1}{n+1} \frac{\partial G_\alpha^m}{\partial y^m} y^i + T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^i \right) \end{aligned} \quad (2.11)$$

To compute (2.11) explicitly, we use the following identities

$$\alpha_{y^k} = \alpha^{-1} y_k, \quad s_{y^k} = \alpha^{-2} (b_k \alpha - s y_k),$$

where $y_i = a_{il} y^l$. Here after, α_{y^k} means $\frac{\partial \alpha}{\partial y^k}$. Then

$$[\alpha Q s_0^m]_{y^m} = \alpha^{-1} y_m Q s_0^m + \alpha^{-2} Q [b_m \alpha^2 - \beta y_m] s_0^m = Q' s_0,$$

and

$$[\Psi(-2Q\alpha s_0 + r_{00}) b^m]_{y^m} = \Psi' \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q\alpha s_0] + 2\Psi [r_0 - Q'(b^2 - s^2) s_0 - Q s s_0]$$

where $r_i = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.10), we have

$$T_{y^m}^m = Q' s_0 + \Psi' \alpha^{-1} (b^2 - s^2) [r_{00} - 2Q\alpha s_0] + 2\Psi [r_0 - Q'(b^2 - s^2) s_0 - Q s s_0]. \quad (2.12)$$

Let F and $\bar{F}$ be two (α, β) -metrics, we assume that they have the same Douglas tensor, i.e.

$$D_{jkl}^i = \bar{D}_{jkl}^i.$$

From (2.8) and (2.11), we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(T^i - \bar{T}^i - \frac{1}{n+1} (T_{y^m}^m - \bar{T}_{y^m}^m) y^i \right) = 0$$

Then there exists a class of scalar function $H_{jk}^i = H_{jk}^i(x)$, such that

$$H_{00}^i = T^i - \bar{T}^i - \frac{1}{n+1} (T_{y^m}^m - \bar{T}_{y^m}^m) y^i, \quad (2.13)$$

where $H_{00}^i = H_{jk}^i y^j y^k$, T^i and $T_{y^m}^m$ are given by (2.10) and (2.12) respectively

III. PROJECTIVE RELATION OF CLASSES OF (α, β) -METRICS

In this section, we find the projective relation between special metric (α, β) -metric

$F = \alpha + \beta - \frac{\beta^2}{\alpha}$ and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ on a same underlying manifold M of dimension $n \geq 3$.

For (α, β) -metric $F = \alpha + \beta - \frac{\beta^2}{\alpha}$, one can prove by (2.3) that F is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$.

The geodesic coefficients are given by (2.4) with

$$\begin{aligned} \theta &= \frac{\{1 + 3s^2 - 4s^3\}}{2\{1 + s - s^2\}\{1 - 2b^2 + 3s^2\}}, \\ Q &= \frac{1 - 2s}{1 + s^2}, \\ \Psi &= -\frac{1}{1 - 2b^2 + 3s^2}, \end{aligned} \quad (3.1)$$

For Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$, the geodesic coefficient are given by (2.4) with

$$\bar{Q} = -\frac{1}{2s}$$

$$\bar{\theta} = -\frac{s}{b^2}$$

$$\bar{\Psi} = \frac{1}{2b^2}. \quad (3.2)$$

In this paper we assume that $\lambda = \frac{1}{n+1}$. Since the Douglas tensor is a projective invariant,

we have,

Theorem 3.2. Let $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an (α, β) - metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n -dimensional manifold $M(n \geq 3)$ where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non zero 1-forms. Then F and $\bar{F}$ have the same Douglas tensors if and only if they are all Douglas metrics.

Proof: First, we prove the sufficient condition.

Let F and $\bar{F}$ be Douglas metrics and corresponding Douglas tensors be D_{jkl}^i and $\bar{D}_{jkl}^i$. Then by the definition of Douglas metric, we have $D_{jkl}^i = 0$ and $\bar{D}_{jkl}^i = 0$, that is both F and $\bar{F}$ have the same Douglas tensor, then (2.13) holds.

Plugging (3.1) and (3.2) into (2.13), we have

$$H_{00}^i = \frac{A^i \alpha^9 + B^i \alpha^8 + C^i \alpha^7 + D^i \alpha^6 + E^i \alpha^5 + F^i \alpha^4 + G^i \alpha^3 + H^i \alpha^2 + I^i}{J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N} + \frac{\bar{A}^i \bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2 \bar{\beta}} \quad (3.3)$$

where

$$\begin{aligned} 2s_0 b^i - 2b^2 s_0^i, & \quad A^i = (1 - 2b^2) \{s_0^i + \\ 2\lambda y^i (r_0 + s_0) - 2\beta s_0^i\}, & \quad B^i = (1 - 2b^2) \{4b^2 \beta s_0^i - 4\beta s_0 b^i - r_{00} b^i + \\ b^2 \lambda s_0 y^i\}, & \quad C^i = \beta [\beta \{ (4b^2 (b^2 - 4) + 7) s_0^i + 4(2 - b^2 s_0 b^i) \} + 4(1 + \\ D^i = \beta [-2\beta^3 \{ (4b^2 (b^2 - 4) + 7) s_0^i + (8 - 4b^2) s_0 b^i \} + (1 + b^2) \lambda s_0 b^i - \beta r_{00} b^i (4b^2 - 5) - \\ 2\lambda y^i \{ 3\beta^2 r_{00} + \beta ((4b^2 - 5) r_0 + (12b^2 - 3) s_0) \}], & \quad E^i = \beta^3 [3\beta \{ 5s_0^i + \\ 2s_0 b^i - 4b^2 s_0^i \} + (4 - 4b^2) s_0 \lambda y^i], & \quad F^i = \beta^3 [6\beta^2 \{ 4b^2 s_0^i - 12s_0 b^i - 5s_0^i \} - (7 - 2b^2) \beta r_{00} b^i + \{ 6(1 - 2b^2) r_{00} + \beta ((14 - \\ (6 - 12b^2) s_0) \} \lambda y^i], & \quad G^i = 9\beta^6 s_0^i, \\ H^i = -3\beta^5 [\beta \{ 6\beta s_0^i + b^i r_{00} \} + 6\lambda y^i \{ (b^2 - 2) r_{00} - \beta (r_0 + s_0) \}], & \quad I^i = 6\beta^7 r_{00} \lambda y^i \end{aligned}$$

And

$$\begin{aligned} J &= (1 - 2b^2)^2, \\ K &= 4\beta^2 (1 - 2b^2) (2 - b^2), \\ L &= 2\beta^4 (11 + 2b^4 - 14b^2), \\ M &= -12\beta^6 (b^2 - 2), \\ N &= 9\beta^8 \end{aligned}$$

And

$$\begin{aligned} \bar{A}^i &= \bar{b}^2 \bar{s}_0^i - \bar{b}^i \bar{s}_0, \\ \bar{B}^i &= \bar{\beta} [2\lambda y^i (\bar{r}_0 + \bar{s}_0) - \bar{b}^i \bar{r}_{00}]. \end{aligned}$$

Further, (3.3) is equivalent to

$$(A^i \alpha^9 + B^i \alpha^8 + C^i \alpha^7 + D^i \alpha^6 + E^i \alpha^5 + F^i \alpha^4 + G^i \alpha^3 + H^i \alpha^2 + I^i) (2\bar{b}^2 \bar{\beta}) + (\bar{A}^i \bar{\alpha}^2 + \bar{B}^i) \times (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) = H_{00}^i (2\bar{b}^2 \bar{\beta}) (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) \quad (3.4)$$

Replacing (y^i) by $(-y^i)$ in (3.4) yields

$$(-A^i \alpha^9 + B^i \alpha^8 - C^i \alpha^7 + D^i \alpha^6 - E^i \alpha^5 + F^i \alpha^4 - G^i \alpha^3 + H^i \alpha^2 + I^i) (-2\bar{b}^2 \bar{\beta}) - (\bar{A}^i \bar{\alpha}^2 + \bar{B}^i) \times (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) = -H_{00}^i (2\bar{b}^2 \bar{\beta}) (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) \quad (3.5)$$

Adding (3.4) and (3.5), we get

$$(A^i \alpha^9 + C^i \alpha^7 + E^i \alpha^5 + G^i \alpha^3)(2\bar{b}^2 \bar{\beta}) = 0$$

Above equation reduces to

$$A^i \alpha^9 + C^i \alpha^7 + E^i \alpha^5 + G^i \alpha^3 = 0 \quad (3.6)$$

Therefore, we conclude that (3.3) is equivalent to

$$H_{00}^i = \frac{B^i \alpha^8 + D^i \alpha^6 + F^i \alpha^4 + H^i \alpha^2 + I^i}{J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N} + \frac{\bar{A}^i \bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2 \bar{\beta}} \quad (3.7)$$

(3.7) is equivalent to

$$\begin{aligned} & (B^i \alpha^8 + D^i \alpha^6 + F^i \alpha^4 + H^i \alpha^2 + I^i)(2\bar{b}^2 \bar{\beta}) + (\bar{A}^i \bar{\alpha}^2 + \bar{B}^i) \times \\ & (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) = H_{00}^i (2\bar{b}^2 \bar{\beta})(J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N) \end{aligned} \quad (3.8)$$

In the above equation (3.8), we can see that $\bar{A}^i \bar{\alpha}^2 (J \alpha^8 + K \alpha^6 + L \alpha^4 + M \alpha^2 + N)$ can be divided by $\bar{\beta}$. Since $\beta = \mu \bar{\beta}$, then $\bar{A}^i \bar{\alpha}^2 J \alpha^8$ can be divided by $\bar{\beta}$. Because $\bar{\beta}$ is prime with respect to α and $\bar{\alpha}$. Therefore $\bar{A}^i = \bar{b}^2 \bar{s}_0^i - \bar{b}^i \bar{s}_0$ can be divided by $\bar{\beta}$. Hence there is a scalar function $\Psi^i(x)$ such that

$$\bar{b}^2 \bar{s}_0^i - \bar{b}^i \bar{s}_0 = \bar{\beta} \Psi^i \quad (3.9)$$

Transvecting (3.9) by $\bar{y}_i = \bar{a}_{ij} y^j$, we get $\Psi^i(x) = -\bar{s}^i$. Thus we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2} (\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i) \quad (3.10)$$

Thus, by lemma 2.3, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metrics. i.e. Both $F = \alpha + \beta - \frac{\beta^2}{\alpha}$,

and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ are Douglas metrics.

If $n = 2$, $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ is a Douglas metric by lemma 2.3. Thus F and $\bar{F}$ have the same Douglas tensors means that they are Douglas metrics. Thus F and $\bar{F}$ have the same Douglas tensors means that they are Douglas metrics. Thus $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ be an special (α, β) -metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric on an n -dimensional manifold $M(n \geq 2)$, where α and $\bar{\alpha}$ are Riemannian metric, β and $\bar{\beta}$ are two non zero collinear 1-forms. Then F and $\bar{F}$ have same Douglas tensors if and only if they are Douglas metrics. This completes the proof of theorem (3.2).

IV. PROOF OF THEOREM 1.1.

First, we prove the necessary condition:

Since Douglas tensor is an invariant under projective changes between two Finsler metrics, If F is projectively related to $\bar{F}$, then they have the same Douglas tensor. According to theorem (3.2), we obtain that both F and $\bar{F}$ are Douglas metrics.

By [3], It is well known that Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ with $b^2 \neq 0$ is a Douglas metric if and only if $s_{ik} = \frac{1}{b^2} (b_i s_k - b_k s_i)$ and also it has been proved that by [7], we know that (α, β) -metric, $F = \alpha + \beta - \frac{\beta^2}{\alpha}$ is a Douglas metric if and only if

$$b_{i|j} = 2\tau \{ (1 - 2b^2) a_{ij} + 3b_i b_j \} \quad (4.1)$$

where $\tau = \tau(x)$ is a scalar function on M . In this case, β is closed.

Plugging (4.1) and (3.1) into (2.4), we have

$$G^i = G_{\alpha}^i + \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau y^i - 2\tau \alpha^2 b^i \quad (4.2)$$

Again plugging (3.10) and (3.2) into (2.4), we have

$$\bar{G}^i = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} \left\{ -\bar{\alpha}^2 \bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00} \bar{b}^i) + 2 \frac{\bar{r}_{00} \bar{\beta} y^i}{\bar{\alpha}^2} \right\} \quad (4.3)$$

Since F is Projectively equivalent to $\bar{F}$, then there exist a scalar function $P = P(x, y)$ on $TM \setminus \{0\}$ such that

$$G^i = \bar{G}^i + P y^i \quad (4.4)$$

By (4.2), (4.3) and (4.4), we have

$$\left[P - \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau - \frac{1}{\bar{b}^2} \left(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2} \right) \right] y^i = G_\alpha^i - \bar{G}_\alpha^i - 2\alpha^2 \tau b^i - \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) \quad (4.5)$$

Note that RHS of above equation is in quadratic form.

Then there must be a one form $\theta = \theta_i y^i$ on M , such that

$$\left[P - \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau - \frac{1}{\bar{b}^2} \left(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2} \right) \right] = \theta$$

Thus (4.5) becomes

$$G_\alpha^i - 2\alpha^2 \tau b^i = \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i \quad (4.6)$$

This completes the proof of necessity.

Conversely from (4.2), (4.3) and (1.5) we have

$$G^i = \bar{G}^i + \left[\theta + \left(\frac{\alpha^3 + 3\alpha\beta^2 - 4\beta^3}{\alpha^2 + \alpha\beta - \beta^2} \right) \tau + \frac{1}{\bar{b}^2} \left(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2} \right) \right] y^i \quad (4.7)$$

Thus F is projectively equivalent to $\bar{F}$. From the above theorem, immediately we get the following corollary

Corollary 4.1. [18]: Let $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ be a special (α, β) -metric and $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ be a Kropina metric be two (α, β) -metrics on a n -dimensional manifold M with dimension $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero collinear 1-forms. Then F is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relations

$$\begin{aligned} G^i - 2\alpha^2 \tau b^i &= \bar{G}_\alpha^i + \frac{1}{2\bar{b}^2} (\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i, \\ s_{ij} &= 0 \\ \bar{s}_{ij} &= \frac{1}{\bar{b}^2} (\bar{b}_i \bar{s}_j - \bar{b}_j \bar{s}_i) \\ b_{ij} &= 2\tau \{ (1 - 2b^2) a_{ij} + 3b_i b_j \} \end{aligned}$$

Where b_{ij} denotes the coefficients of the covariant derivative of β with respect to α .

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